MATHEMATICS

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AIM POINT
MATHEMATICS
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BINOMIAL THEOREM

& Their Properties

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THINGS TO REMEMBER

✱ <u>Bionomial Theorem (For Positive integer)</u>

If n is a positive integer and x, a $\in$ R, then

$(x + a)n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \ldots\ldots. + {}^nC_nx^n$

and $(x - a)n = {}^nC_0x^n - {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 - \ldots\ldots. + (-1)^n a^n$

Coefficients ${}^nC_0$, ${}^nC_0$, ${}^nC_0$,............, ${}^nC_0$, are known as binomial coefficients

and $${}^nC_r = \frac{n!}{r!(n-r)!}.$$

General Term of Binomial Theorem

Let the general term in the expansion of $(x + a)n$ is (r + 1)th term.

$$T_{r+1} = {}^nC_rx^{n-r}a^r$$

Properties of Expansion of Binomial Theorem

1. The sum of indices of x and a in each term is n.
2. The number of terms in $(x + a)^n$ is (n + 1).
3. The coefficient of terms equidistant from the beginning and the end are equal.
4. (i) $(x + a)^n + (x - a)^n = 2[{}^nC_0x^n + {}^nC_0x^{n-2}a^2 + \ldots\ldots\ldots..]$

   (ii) $(x + a)^n - (x - a)^n = 2[{}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \ldots\ldots\ldots..]$
5. If n is odd, then $\{(x + a)^n + (x - a)^n\}$ and $\{(x + a)^n - (x - a)^n\}$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$, whereas if n is even, then $\{(x + a)^n + (x - a)^n\}$ has $\left(\frac{n+1}{2}\right)$ terms and $\{(x + a)^n - (x - a)^n\}$ has $\frac{n}{2}$ terms.
6. In the binomial expansion of $(x + a)^n$, rth term from the end is (n – r + 2)th term from the beginning.

✱ <u>Middle Term in Binomial Expansion</u>

1. If n is an even number, then $\left(\frac{n}{2}+1\right)$th term is middle term in the expansion of $(x + a)^n$.

   $$\therefore \qquad T_{\frac{n}{2}+1} = {}^nC_{n/2}x^{n/2}a^{n/2}$$

2. If n is an odd number, then $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+1}{2}\right)$th terms are middle terms middle terms in the expansion of $(x + a)^n$.

   $$\therefore \qquad T_{\frac{n+1}{2}} = {}^nC_{\frac{n-1}{2}}x^{\frac{n+1}{2}}a^{\frac{n-1}{2}}$$

   and $$T_{\frac{n+3}{2}} = {}^nC_{\frac{n+1}{2}}x^{\frac{n-1}{2}}a^{\frac{n+1}{2}}$$

✷ Greatest Term in the Expansion of $(x + a)^n$

If T_r and T_{r+1} be the rth and (r + 1)th terms in the expansion of $(x + a)^n$, then

$$\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r x^{n-r} a^r}{{}^nC_{r-1} x^{n-r+1} a^{r-1}}\right|$$

$$= \left(\frac{n-r+1}{r}\right)\left|\frac{a}{x}\right|$$

$\because \qquad \dfrac{T_{r+1}}{T_r} \geq 1 \qquad \Rightarrow \qquad \dfrac{n-r+1}{r}\left|\dfrac{a}{r}\right| \geq 1$

$\Rightarrow \qquad (n - r + 1)\,|a| \geq r\,|x|$

$\Rightarrow \qquad r \leq \dfrac{(n+1)|a|}{|x|+|a|}$

Let $\qquad \dfrac{(n+1)|a|}{|x|+|a|} = I + f$

(Where I is an integer and $0 \leq f < 1$)

If $f = 0$, then T_r and T_{r+1} are equal and greatest and if $0 < f < 1$ then T_{r+1} will be greatest.

Properties of Binomial Coefficients

1. If ${}^nC_r = {}^nC_s$, then either r = s or r + s = n
2. ${}^nC_0 + {}^nC_1 + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n$
3. ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots\ldots = {}^nC_1 + {}^nC_3 + \ldots\ldots\ldots = 2^{n-1}$
4. ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots\ldots + (-1)^n\,{}^nC_n = 0$
5. $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$
6. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
7. ${}^{n+1}C_{r+1} = \dfrac{n+1}{r+1}\cdot{}^nC_r$
8. ${}^nC_r + 2\,{}^nC_r + 3\,{}^nC_r + \ldots\ldots\ldots\ldots + n\,{}^nC_n = n \cdot 2^{n-1}$
9. ${}^nC_1 - 2\,{}^nC_2 + 3\,{}^nC_3 - \ldots\ldots = 0$
10. ${}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots\ldots + (n + 1)\,{}^nC_n = (n + 2)2^{n-1}$
11. $C_0C_r + C_1C_{r+1} + \ldots\ldots\ldots + C_{n-r}C_n = \dfrac{(2n)!}{(n-r)!(n+r)!}$
12. $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots\ldots + C_n^2 = \dfrac{(2n)!}{(n!)^2}$
13. $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots\ldots = \begin{cases} 0 & , \\ (-1)^{n/2}\,{}^nC_{n/2} & , \end{cases}$

14. $C_1^2 - 2C_2^2 + 3C_3^2 - ......+ (-1)^n nC_n^2 = (-1)^{\frac{n}{2}-1} . \frac{n}{2} . \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}$, where n is even.

✷ Binomial Theorem (For Any Exponent)

Let n is a rational number and x is a real number such that $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

eg, The expansion $(2 + 3x)^{-5}$ upto four terms in decreasing power of x, is as follows

$$(2+3x)^{-5} = \left[3x\left(1+\frac{2}{3x}\right)\right]^{-5}$$

$$= \frac{1}{243x^5}\left[1 + (-5)\left(\frac{2}{3x}\right) + \frac{(-5)(-6)}{2!}\left(\frac{2}{3x}\right)^2 + \frac{(-5)(-6)(-7)}{3!}\left(\frac{2}{3x}\right)^3 + ...\right]$$

$$= \frac{1}{243x^5}\left[1 - \frac{10}{3x} + \frac{20}{3x^2} - \frac{280}{27x^3} + ...\right]$$

$$= \frac{1}{243}\left[\frac{1}{x^5} - \frac{10}{3}.\frac{1}{x^6} + \frac{20}{3}.\frac{1}{x^7} - \frac{280}{27}.\frac{1}{x^8} + ...\right]$$

General Term in the Expansion of $(1 + x)^n$

General term in the expansion of $(1 + x)^n$ is as follows

$$T_{r+1} = \frac{\{n(n-1)(n-2).......(n-(r-1))\}}{r!}x^r$$

Important Expansions

1. $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + ...+ \frac{(-1)^r n(n+1)(n+2).....(n+r+1)}{r!}x^r + ...$

2. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ...+ \frac{n(n+1)(n+2).....(n+r-1)}{r!}x^r + ...$

3. $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - ....+ (-1)^r \frac{n(n-1)(n-2).....(n-r+1)}{r!}x^r + ...$

4. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ... + (-1)^r x^r + ...$
5. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + ... + x^r +.......$
6. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... + (-1)^r (r + 1)x^r +........$
7. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... + (r + 1)x^r + ...$
8. $(1 + x)^{-3} = 1 - 3x + 6x^2 - ....$
9. $(1 - x)^{-3} = 1 + 3x + 6x^2 + ...$

✵ Multinomial Theorem

If n is a positive integer and $a_1, a_2, \dots\dots, a_k \in R$, then $(a_1 + a_2 + \dots\dots + a_k)^n = \sum_{r_1+r_2+\dots\dots+r_k=n} \frac{n!\, a_1^{r_1} \cdot a_2^{r_2} \dots\dots a_k^{r_k}}{r_1!\, r_2! \dots\dots r_k!}$

Note :

- General term in the expansion of $(x - a)^n$ is

  $$T_{r+1} = (-1)r \,.\, {}^nC_r\, x^{n-r} a^r$$

- General term in the expansion of $(1 - x)^n$ is

  $$T_{r+1} = {}^nC_r\, x^r$$

- General term in the expansion of $(1 - x)^n$ is

  $$T_{r+1} = (-1)^r\, {}^nC_r\, x^r$$

- When there are two middle terms in the expansion, then their binomial coefficients are equal.
- Binomial coefficient of middle term is the greatest binomial coefficient.
- If the value of x is so small than on leaving square and higher powers of x, we get.

  $$(1 + x)^n = 1 + nx$$

- The number of terms in $(x + y + z)^n$ is ${}^{n+2}C_2$.
- The number of terms in the expansion of (a1 + a2 + ... + ak)n is ${}^{n+k-1}C_{k-1}$.
- Coefficient of xn1. yn2. zn3 in the expansion of $(x + y + z)^n$ is $\frac{n!}{n_1!\, n_2!\, n_3!}$ where $n = n_1 + n_2 + n_3$.
- Sum of coefficients in the expansion of $(a + bx + cx^2)^n$ is $(a + b + c)^n$.